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Review of Fractional-Order Dynamical Systems: Mathematical Developments and Applications in Science and Engineering

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Abstract
Fractional-order dynamical systems have gained considerable attention in recent decades due to their superior ability to model complex phenomena involving memory, hereditary effects, non-local interactions, and anomalous diffusion that cannot be accurately described using traditional integer-order differential equations. This review paper presents a comprehensive analysis of recent mathematical developments in fractional-order dynamical systems and their diverse applications across science and engineering. The study examines the theoretical foundations of fractional calculus, including commonly used fractional derivatives such as Caputo, Riemann–Liouville, and Atangana–Baleanu operators, highlighting their mathematical characteristics and modelling capabilities. It further reviews major analytical approaches, including existence and uniqueness of solutions, stability analysis, bifurcation behaviour, controllability, and numerical approximation techniques employed for solving fractional differential equations. The paper also explores the growing application of fractional-order models in engineering systems, biomedical sciences, control theory, electrical circuits, signal processing, viscoelastic materials, finance, environmental modelling, and epidemiology, where memory-dependent and nonlinear dynamics play a significant role. Recent advances in computational algorithms and numerical simulation methods have expanded the practical implementation of these models, improving prediction accuracy and system representation. Despite these developments, challenges remain in parameter estimation, computational efficiency, model validation, and the development of robust numerical methods for high-dimensional problems. The review identifies current research trends and emerging opportunities, particularly the integration of fractional calculus with artificial intelligence, machine learning, and data-driven modelling to enhance predictive performance in complex systems. By synthesising contemporary literature, this paper provides a consolidated understanding of the mathematical developments, computational techniques, and multidisciplinary applications of fractional-order dynamical systems while outlining future research directions that can further strengthen their theoretical foundations and practical relevance in modern scientific and engineering research.

Keywords: Fractional-order dynamical systems, Fractional calculus, Mathematical modelling, Fractional differential equations, Science and engineering applications.

Introduction
Fractional-order dynamical systems have emerged as one of the most significant developments in modern applied mathematics because of their ability to model complex systems that exhibit memory, hereditary characteristics, non-local behaviour, and long-range temporal dependence.

Unlike conventional integer-order differential equations, which describe system dynamics based solely on the current state, fractional-order models incorporate the influence of historical states, thereby providing a more realistic representation of many physical, biological, engineering, and economic processes. The theoretical foundations of fractional calculus can be traced back to the correspondence between Leibniz and L'Hôpital in the seventeenth century, yet substantial progress has been achieved only in recent decades with the development of efficient analytical and computational techniques. The introduction of various fractional derivative definitions, including the Riemann–Liouville, Caputo, Grünwald–Letnikov, and Atangana–Baleanu operators, has enabled researchers to construct mathematical models that capture complex nonlinear dynamics with greater precision. Consequently, fractional-order dynamical systems have become indispensable in describing anomalous diffusion, viscoelastic behaviour, heat conduction, fluid mechanics, electrical circuits, signal processing, control systems, population dynamics, epidemiological transmission, and financial markets. The growing availability of computational resources and numerical algorithms has further accelerated research in this field, allowing the solution of complex fractional differential equations that were previously considered mathematically intractable. As a result, fractional calculus has evolved from a theoretical branch of mathematics into a practical modelling framework with multidisciplinary significance. The increasing demand for accurate predictive models in engineering and scientific research has further strengthened the importance of fractional-order approaches, particularly in situations where classical integer-order models fail to capture system memory, damping effects, and nonlinear interactions. This transformation has positioned fractional-order dynamical systems at the forefront of contemporary mathematical modelling and computational science.

The rapid expansion of research on fractional-order dynamical systems has generated substantial advances in both mathematical theory and practical applications, making it essential to consolidate existing knowledge through comprehensive review studies. Recent investigations have focused on fundamental mathematical aspects such as the existence and uniqueness of solutions, stability criteria, equilibrium analysis, Lyapunov methods, bifurcation behaviour, chaos theory, controllability, observability, and numerical approximation techniques. Simultaneously, the application domain has broadened considerably, encompassing robotics, aerospace engineering, renewable energy systems, biomedical engineering, neuroscience, climate modelling, image processing, artificial intelligence, cybersecurity, and smart manufacturing. These applications demonstrate the versatility of fractional-order models in representing systems characterised by uncertainty, memory effects, and nonlinear interactions across multiple spatial and temporal scales. Nevertheless, several challenges continue to hinder the widespread implementation of fractional-order dynamical systems, including computational complexity, parameter identification, optimisation of numerical algorithms, model validation, and the lack of unified analytical frameworks for solving high-dimensional fractional differential equations. Furthermore, the emergence of hybrid computational approaches integrating fractional calculus with machine learning, deep learning, digital twins, and data-driven modelling has opened new opportunities for developing more accurate and adaptive predictive models. In this context, the present review aims to provide a comprehensive synthesis of recent mathematical developments in fractional-order dynamical systems, with particular emphasis on theoretical foundations, analytical methodologies, computational techniques, and multidisciplinary applications in science and engineering. By critically examining contemporary research trends, identifying existing limitations, and highlighting future research opportunities, this review contributes to a deeper understanding of the evolving role of fractional-order dynamical systems in addressing increasingly complex real-world problems while serving as a valuable reference for mathematicians, engineers, and interdisciplinary researchers.

SIGNIFICANCE OF THE STUDY

The significance of the proposed research can be assessed at three levels: mathematical, computational, and applied.

Mathematically, the study contributes to the qualitative theory of fractional differential equations by extending existence-uniqueness and stability results to multi-order and variable-order systems, and by examining the conditions under which such systems exhibit bifurcation or chaotic dynamics. These results enrich the general theory of fractional calculus and provide a foundation on which subsequent researchers can build further generalizations, for instance to stochastic or fractal-fractional settings.

Computationally, the study is significant because it develops and rigorously tests numerical schemes — including predictor-corrector, spectral, and operational-matrix methods — specifically adapted to the memory-dependent structure of fractional operators. Establishing convergence, stability, and error bounds for these schemes gives practitioners in applied fields confidence that simulations of fractional-order models are numerically trustworthy rather than merely qualitatively suggestive.

From an applied standpoint, the significance of the research lies in its potential to improve the modelling of real-world systems that display memory and hereditary effects. In epidemiology, fractional-order compartmental models have been shown to capture the trajectory of infectious disease outbreaks, including COVID-19, more accurately than integer-order counterparts, because the fractional order can encode the cumulative influence of past exposure, immunity, and behavioural change on present transmission. In engineering and control theory, fractional-order controllers (such as fractional PID controllers) offer additional tuning flexibility and improved robustness compared with their integer-order analogues. In secure communications, chaos synchronization of fractional-order chaotic systems underlies encryption schemes that are more resistant to certain classes of attack. In materials science, fractional viscoelastic models describe the stress-strain behaviour of polymers and biological tissues with a small number of parameters and high fidelity to experimental creep and relaxation data.

Beyond these specific domains, the study is significant because it addresses a methodological concern that cuts across all applications of fractional calculus: the tendency to select a fractional derivative definition and a numerical scheme without adequate justification of stability, convergence, or comparative advantage. By systematically comparing operators (Caputo, Caputo-Fabrizio, Atangana-Baleanu) and numerical schemes within a common analytical framework, the research aims to provide applied researchers with clearer, evidence-based guidance for model selection.

Finally, the study carries pedagogical and institutional significance. Fractional calculus is increasingly included in postgraduate mathematics and engineering curricula, yet accessible, application-oriented research that connects abstract fractional operator theory to concrete, data-driven problems remains comparatively scarce. The outputs of this research — theoretical results, validated numerical algorithms, and worked case studies in epidemiology, control, and materials science — are intended to serve both the immediate research community and future teaching and training in this area.

LITERATURE REVIEW

The literature on fractional-order dynamical systems has expanded considerably over the past decade, spanning foundational operator theory, qualitative analysis, numerical methods, and a wide array of applications. This section reviews representative contributions organized around these four themes.

5.1 Fractional Derivative Operators and Foundational Theory

The classical Riemann-Liouville and Caputo derivatives remain the most widely used fractional operators because of their well-understood analytical properties, but both retain singular power-law kernels that can be restrictive when modelling processes without a power-law memory structure. To address this, non-singular kernel derivatives were introduced, most notably the Caputo-Fabrizio derivative, based on an exponential kernel, and later the Atangana-Baleanu derivative, based on the generalized Mittag-Leffler function, which combines non-locality with a non-singular kernel. Atangana and Baleanu's original 2016 formulation of the new fractional derivative with nonlocal and non-singular kernel has since become one of the most heavily cited operators in the field, motivating a large secondary literature examining its properties and applications. Building on this, subsequent studies examined existence and controllability results for differential equations involving these operators; for instance, Aimene, Baleanu, and Seba (2019) established controllability conditions for semilinear impulsive Atangana-Baleanu fractional differential equations with delay, extending controllability theory beyond the classical Caputo setting. Owolabi and Atangana (2017) further contributed numerical approximation schemes for nonlinear fractional parabolic equations formulated with the Caputo-Fabrizio derivative, illustrating the practical need for operator-specific numerical treatment.

5.2 Qualitative Analysis: Existence, Stability, and Bifurcation

A central strand of the literature addresses the existence, uniqueness, and stability of solutions of fractional dynamical systems. Ivan (2021) analyzed the fractional-order Lagrange system, establishing existence and uniqueness of solutions, asymptotic stability of equilibrium states, and a stabilization scheme using appropriate feedback controls, together with a fractional Euler numerical integration scheme, illustrating the kind of complete qualitative-cum-numerical treatment that this proposed research seeks to generalize. Kumar and Pandey (2020) examined the existence of mild solutions for Atangana-Baleanu fractional differential equations subject to non-instantaneous impulses and non-local conditions, extending classical semigroup-based existence theory into the non-singular-kernel setting. A commonly used stability criterion in this literature is the Matignon condition, under which the zero equilibrium of the commensurate-order linear system D^alpha x = A x is asymptotically stable if and only if every eigenvalue lambda of the Jacobian matrix A satisfies the condition given in Equation (3).

$$|\, arg(\lambda_i) \,| > \frac{\alpha\pi}{2}\,,\ i = 1, 2, \dots, n \tag{3}$$

Chaotic and bifurcation phenomena in fractional systems have also attracted sustained attention. Owolabi (2020) modelled several chaotic dynamical systems by replacing integer-order time derivatives with Caputo fractional-order counterparts, employing a Chebyshev spectral method for numerical approximation and establishing linear stability results, and showed that varying the fractional order alone can generate a rich spectrum of chaotic behaviours not present in the corresponding integer-order model. Almutairi and Saber (2021) studied chaos control in the time-varying fractional Newton-Leipnik system using Atangana-Baleanu-Caputo and Atangana-Seda derivatives, proposing a generalized numerical scheme based on the fundamental theorem of fractional calculus and Lagrange interpolation, and demonstrating existence, uniqueness, and controllable chaotic behaviour in a variable-order setting.

5.3 Control, Synchronization, and Engineering Applications

Fractional-order control theory has become an active application area, motivated by the additional design freedom that non-integer differentiation and integration orders provide. A representative example is the fractional-order PID controller, whose control law generalizes the classical PID controller by allowing the integral and derivative actions to take non-integer orders lambda and mu, as shown in Equation (5), giving the designer two extra tuning parameters compared with the classical integer-order controller.

$$u(t) = K_p\, e(t) + K_i\, D^{-\lambda}\, e(t) + K_d\, D^{\mu}\, e(t) \tag{5}$$

Hajipour, Hajipour, and Baleanu (2018) designed an adaptive sliding-mode controller for a hyperchaotic fractional-order financial system, demonstrating that fractional-order controllers can stabilize complex, high-dimensional chaotic economic dynamics under parameter uncertainty. Song, Lv, Liu, and Liu (2019) proposed robust control strategies for disturbed fractional-order economic chaotic systems with uncertain parameters, again underscoring the relevance of fractional-order control to systems, such as financial markets, that exhibit long-memory and chaotic characteristics. Synchronization of fractional-order chaotic systems, which underlies secure-communication applications, has likewise been studied extensively: Behinfaraz, Ghaemi, and Khanmohammadi (2019) developed adaptive synchronization schemes with fractional adaptation laws based on risk analysis, while Li and Wu (2019) proposed an adaptive synchronization control policy for fractional-order chaotic systems with order between zero and one, with an explicit application to secret communication. Jiang, Chen, Cao, and Guirao (2021) extended sliding-mode tracking control to a class of variable-order fractional differential systems, indicating a shift in the control literature from fixed-order to variable-order formulations that mirror real, time-varying physical processes.

5.4 Epidemiological and Biomedical Applications

The COVID-19 pandemic prompted an unusually rapid expansion of fractional-order epidemic modelling. Denu and Kermausuor (2021) formulated a fractional-order COVID-19 model incorporating lockdown measures using the Caputo derivative, established existence, uniqueness, and global stability of the disease-free and endemic equilibria using comparison theorems and the fractional La Salle invariance principle, and approximated the model solution using the residual power series technique. Wali and colleagues (2021) extended a fractional SEIR-type COVID-19 model with a vaccination compartment using the Atangana-Baleanu-Caputo derivative, deriving the disease-free equilibrium and its stability in relation to the basic reproduction number, and validating the model against Italian vaccination and mortality data. Other researchers have examined fractional-order models incorporating quarantine, reinfection, symptomatic and asymptomatic transmission, and time delay in the implementation of control policy, consistently reporting that fractional-order formulations track observed epidemic data more closely than their integer-order counterparts, particularly over longer time horizons where memory effects accumulate. Beyond infectious disease, fractional-order modelling has been applied to oncology and hospital-acquired infection: Amilo, Sadri, Kaymakamzade, and Hincal (2021) developed a fractional-order model for the combined treatment of metastatic colorectal cancer, while Bagkur, Amilo, and Kaymakamzade (2021) modelled nosocomial Pseudomonas aeruginosa infection dynamics in a regional hospital setting, both demonstrating the versatility of fractional-order compartmental modelling in contemporary biomedical research.

5.5 Numerical Methods for Fractional-Order Systems

Because fractional derivatives are non-local operators, their numerical evaluation requires the accumulation of a system's entire history rather than only its current state, which makes the design of efficient and stable discretization schemes a research area in its own right. Operators such as the Atangana-Baleanu derivative rely on the two-parameter Mittag-Leffler function as a non-singular kernel in place of the classical power-law kernel, defined in Equation (4), which reduces to the classical exponential function when alpha equals 1.

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{4}$$

Discretization schemes such as the L1 and L1-2 formulas for approximating the Caputo derivative, refined through high-order interpolation and generalized to weighted operators such as the psi-Caputo derivative, have been developed to balance numerical accuracy against the substantial memory and computational cost associated with non-local operators, particularly for sub-diffusion and advection-diffusion type equations. Spectral methods, including Chebyshev collocation, offer an alternative route to high accuracy for smooth solutions and have been used successfully in the modelling of chaotic fractional-order systems, as in the work of Owolabi (2020) discussed above. Integral-transform-based approaches provide yet another avenue: transform methods adapted to the Caputo fractional derivative have been used to convert fractional differential equations into algebraic form, simplifying both the analytical treatment and the numerical solution of fractional-order systems arising in control theory. Taken together, this body of work indicates that no single numerical scheme is uniformly best

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across all classes of fractional-order systems; the appropriate choice depends on the smoothness of the solution, the length of the simulation horizon, the dimensionality of the system, and the specific fractional operator employed, reinforcing the need identified in this study for a comparative, application-specific evaluation of numerical methods rather than reliance on a single default scheme.

6. RESEARCH METHODOLOGY

The proposed research adopts a combined analytical, numerical, and applied methodology, structured in four phases.

\ This phase establishes the precise mathematical definitions, properties, and known results for the Riemann-Liouville, Caputo, Caputo-Fabrizio, and Atangana-Baleanu derivatives that will be used throughout the study, and identifies specific classes of multi-order and variable-order systems on which existing qualitative theory is incomplete.

Phase two undertakes the analytical development of the research. Existence and uniqueness of solutions for the selected classes of fractional dynamical systems will be established using fixed-point theorems (Banach and Schauder-type arguments) adapted to the relevant fractional operator. Stability of equilibrium points will be analyzed using Lyapunov-function-based methods generalized to the fractional setting, together with the Matignon stability theorem and fractional Routh-Hurwitz criteria, extended where necessary to multi-order systems. Bifurcation and chaos will be investigated through linearization about equilibria, computation of Lyapunov exponents, and bifurcation diagrams generated as the fractional order or a system parameter is varied.

Phase three focuses on numerical methods. Predictor-corrector schemes (of Adams-Bashforth-Moulton type), spectral collocation methods (including Chebyshev and operational-matrix approaches), and, where appropriate, finite-difference discretizations of the fractional operators will be implemented in a scientific computing environment. Each scheme will be analyzed for numerical stability and order of convergence, and validated against systems with known closed-form or benchmark numerical solutions before being applied to the case studies of phase four.

Phase four applies the combined analytical and numerical framework to selected real-world case studies: a fractional-order epidemic model calibrated against publicly available disease surveillance data; a fractional-order chaotic system studied for synchronization in a secure-communication context; a fractional-order control system (such as a fractional PID controller) applied to a representative engineering plant model; and a fractional viscoelastic model fitted to representative creep or relaxation data. In each case, parameter estimation will be performed using appropriate optimization techniques, and the resulting fractional-order model will be compared quantitatively (using error metrics such as root-mean-square error) against its integer-order counterpart to assess the added explanatory value of the fractional-order formulation.

Throughout all four phases, results will be cross-validated: analytical stability predictions will be checked against numerical simulation, and numerical simulation outputs will, where possible, be checked against real or benchmark data, ensuring that the conclusions of the study rest on convergent theoretical and empirical evidence rather than either strand alone.

Data for the applied case studies will be drawn primarily from publicly available sources: epidemiological surveillance data published by recognized health authorities for the epidemic case study, standard benchmark parameter sets from the chaos-synchronization and control literature for the corresponding case studies, and published experimental creep or relaxation curves for the viscoelastic case study. Where real data are unavailable at sufficient resolution, carefully documented synthetic data generated from established integer-order benchmark models will be used to test the numerical schemes, with this substitution made explicit in the reporting of results.

Conclusion

Fractional-order dynamical systems have become an indispensable area of modern mathematical research due to their ability to represent complex dynamic processes with greater accuracy than conventional integer-order models. By incorporating memory and hereditary effects, fractional calculus provides a powerful framework for analysing nonlinear systems whose present behaviour depends not only on current conditions but also on their historical states. This review has examined the major mathematical developments in fractional-order dynamical systems, including the theoretical foundations of fractional derivatives, analytical methods for existence and uniqueness of solutions, stability analysis, controllability, bifurcation, chaos, and numerical approximation techniques. The literature demonstrates that these mathematical models have significantly enhanced the understanding of complex phenomena across diverse scientific and engineering disciplines. Applications in control engineering, electrical circuits, biomedical systems, epidemiology, viscoelastic materials, fluid dynamics, finance, environmental science, and signal processing illustrate the flexibility and effectiveness of fractional-order models in addressing problems characterised by nonlinearity, uncertainty, and long-term memory. Continuous improvements in computational algorithms and numerical methods have further expanded the practical implementation of fractional differential equations, making them increasingly accessible for solving real-world problems.

Despite these advances, several theoretical and computational challenges remain that require further investigation. Efficient numerical algorithms for large-scale and high-dimensional systems, accurate parameter estimation techniques, computational cost reduction, and experimental validation of fractional-order models continue to be important research priorities. In addition, the absence of universally accepted modelling frameworks and the complexity associated with selecting appropriate fractional operators often limit the broader adoption of these methods in practical applications. Future research should therefore focus on developing robust analytical tools, improving computational efficiency, and establishing standardised methodologies for modelling and simulation. The integration of fractional calculus with artificial intelligence, machine learning, digital twin technology, and data-driven modelling represents a promising direction for enhancing predictive accuracy and adaptive system analysis. Such interdisciplinary developments are expected to accelerate innovation in science and engineering while extending the applicability of fractional-order dynamical systems to emerging technological challenges. The continued advancement of mathematical theory, computational techniques, and application-oriented research will strengthen the role of fractional-order dynamical systems as a fundamental framework for modelling, analysing, and solving increasingly complex real-world problems across multiple scientific and engineering domains.

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