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Analyzing Higher-Order Boundary Value Problems within the Framework of Time Scales

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Abstract
This research focuses on higher-order boundary value problems (BVPs) for delay differential equations, a topic of growing significance in control theory, physics andapplied mathematics. While lower-order BVPs with delays have been widely explored, higher-order cases remain comparatively underexamined. This study aims to bridge that gap by presenting a clear first-order system representation and developing multi-point higher-order BVPs for ordinary differential equations. The paper explores the intricacies of two-point and multi-point boundary conditions, emphasizing the role of distinct boundary constraints and the periodic reformulation of systems to enhance computational efficiency. It critically investigates the fundamental issues of existence and uniqueness of solutions, acknowledging the theoretical challenges these problems present. In addition, the study provides a concise review of numerical approaches such as collocation methods, shooting techniques andfinite difference schemes. As the analysis unfolds, it becomes evident that higher-order delayed BVPs pose complex theoretical and computational challenges, offering fertile ground for further research and refinement of numerical strategies. Ultimately, this work contributes to a deeper understanding of boundary value problems on time scales and lays a strong foundation for future advancements in mathematical modeling, computational methods anddynamic system analysis.
Keywords: Delay Differential Equations, Time Scales Calculus, Multi-point BVPs, Ordinary Differential Equations, Explicit First-order System, Mathematical Modeling

INTRODUCTION

Boundary value problems (BVPs) for delay differential equations play a crucial role across a wide range of disciplines, including applied mathematics, control theory, variational problems andmaterials science. In recent years, significant progress has been made in studying lower-order BVPs with delays, yielding promising results and insights. However, higher-order BVPs have received comparatively little attention despite their broad potential applications. For higher-order delay differential equations, BVPs offer a rich framework for modeling complex systems across multiple fields. Notably, multi-point higher-order BVPs for ordinary differential equations have only recently begun to be explored, marking an important step toward expanding the theoretical and practical understanding of such problems.

Overview of Boundary Value Problems (BVPs)

Boundary value problems (BVPs) are a class of ordinary differential equations (ODEs) defined by constraints at the boundaries rather than initial conditions. Unlike initial value problems, a BVP may have no solution, an infinite number of solutions, or a finite set of solutions, depending on the problem's structure and boundary conditions. A crucial step in solving a BVP is to provide an initial guess for the solution. The accuracy of this initial estimate can significantly influence both the performance of the numerical solver and the quality of the final solution. Modern solvers such as bvp4c and bvp5c are well-equipped to handle a wide variety of boundary value problems, including those with two-point or multi-point boundary conditions, singularities, or uncertain (fuzzy) boundaries. These tools enable robust and flexible computation of solutions in complex boundary value settings.

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A two-point boundary value problem (BVP) of order n on a finite span [a,b] can be expressed as an explicit first-order system of ordinary differential equations (ODEs) when the boundary values are evaluated at two different locations.

$$y'(x) = f(x, y(x)), \quad x \in (a, b), \qquad g(y(a), y(b)) = 0 \tag{1}$$

Here, $y, f, g \in R^n$ Since the subsidiary y′ shows up straightforwardly, the system is alluded to as explicit. When the function ggg is linear, the corresponding boundary conditions must also be linearly independent. Moreover, the nnn boundary conditions defined by ggg should be mutually independent, meaning they cannot be resolved separately without affecting one another.

Typically, boundary value problems (BVPs) are not derived directly from a single, compact equation. Instead, they usually originate from a system of equations involving various orders of derivatives, summing to a total of nnn. In an explicit BVP formulation, boundary conditions and right-hand side expressions of the ordinary differential equations (ODEs) may depend on solution variables up to a certain derivative order—but not necessarily on the highest-order derivative featured on the left-hand side of the equation. To manage this complexity, we often define a vector yyy that includes all solution variables and their derivatives up to a certain order, excluding the highest-order terms. This allows the construction of an equivalent first-order system by introducing additional ODEs to represent these derivatives. However, such reformulated systems are generally not optimal for computational purposes and may lack numerical efficiency or precision. In contrast to initial value problems (IVPs), where boundary conditions are specified at a single point, the boundary condition function ggg in BVPs is evaluated at both endpoints of the interval, aaa and bbb. This is why such problems are referred to as two-point boundary value problems. In more complex cases, conditions may be specified at multiple points within the interval (a,b)(a, b)(a,b), resulting in a multipoint BVP. To address a multipoint BVP, one common approach is to partition the interval and define separate solution variables for each subinterval. Additional boundary conditions are then imposed to ensure continuity and consistency across the entire domain. However, reducing a multipoint BVP to a two-point form by such means often leads to a restructured but suboptimal formulation, which may not yield the most efficient or accurate computational solution.

Most of two-point BVPs that emerge practically speaking have unmistakable boundary conditions, implying that the capability g can be isolated into two parts (one for every endpoint):

$$g_a(y(a)) = 0, \qquad g_b(y(b)) = 0.$$

Here, $g_a \in R^s$ and $g_b \in R^{n-s}$ where every one of the vector capabilities ga and gb is autonomous andfor some value s where 1<s<n. Regardless, there are notable, every now and again happening boundary conditions that are not isolated; take intermittent boundary conditions, for example, which are as per the following for a situation based issue:

LITERATURE REVIEW

Bohner and Peterson's (2001) a significant advancement in the study of time-dependent equations, *Dynamic Equations on Time Scales: A Presentation with Applications* offers a novel and unified approach to analyzing both discrete and continuous systems. The book bridges the gap between discrete and continuous calculus by presenting a comprehensive examination of dynamic equations and their applications. The authors introduce time scale calculus, a versatile framework that encompasses discrete and continuous calculus as special cases. This pioneering work equips researchers and practitioners with a powerful analytical tool, establishing a solid foundation for studying dynamic processes across diverse time scales and broadening the scope of mathematical modeling in various scientific and engineering disciplines.

Agarwal, Bohner andPeterson (2009) Published in *Mathematical Inequalities & Applications*, the article "Inequalities on Time Scales: A Survey" makes a valuable contribution to the evolving field of time-scale mathematics. This comprehensive survey addresses inequalities on time scales, highlighting their significance and offering an extensive overview of existing research. The authors explore various aspects of inequalities, considering both continuous and discrete scenarios within the unified framework of time scales. By synthesizing key developments and results, the survey serves as a vital resource for researchers interested in exploring the intricate connections between inequality theory and time-scale calculus, thereby fostering a deeper understanding of this emerging area of mathematics.

Sun and Wang (2013) the article "Existence of Positive Solutions for Higher-Order Solitary Boundary Value Problems on Time Scales," published in *Advances in Difference Equations*, marks a significant advancement in the application of time scale calculus to boundary value problems. This study investigates the existence of positive solutions for higher-order solitary boundary value problems, extending beyond the traditional continuous and discrete frameworks. By employing time scale methods, the authors offer a unified approach that bridges these domains. Their findings not only underscore the effectiveness of time scale calculus in addressing complex, real-world problems but also contribute meaningfully to the growing body of research on dynamic equations and boundary value analysis within the time scales context.

Anderson and Avery (2010) this study centers on impulsive boundary value problems associated with higher-order differential equations on time scales, offering a detailed examination of impulsive effects within the framework of time-scaled dynamic systems. The authors investigate how impulses influence the behavior of solutions to higher-order differential equations, highlighting the distinct features that arise when these problems are approached through the lens of time scale calculus. Their findings contribute to a deeper understanding of the dynamics of systems influenced by impulsive forces, with broad implications for applications across various scientific and engineering disciplines.

Karpuz (2015) this work enhances the study of nonlinear boundary value problems on time scales, particularly those involving higher-order dynamic equations. It delves into the challenges posed by nonlinearity within the context of dynamic systems evolving on time scales, building upon the foundational contributions of Anderson and Avery (2010). Karpuz's research offers significant insights into the existence and uniqueness of solutions for nonlinear higher-order dynamic equations, emphasizing the critical role of time scale analysis in addressing boundary value problems. This contribution further strengthens the theoretical framework and applicability of time scale calculus in complex dynamical systems.

Pečarić and Perić (2006) the authors conduct an analysis of Hermite-Hadamard-type inequalities for sss-convex functions, extending the classical inequality to the context of time scales. Their study broadens the scope of the Hermite-Hadamard inequality by adapting it to sss-convex functions defined on unified temporal domains, reflecting the growing significance of time scale calculus in modern mathematical analysis and its applications. Through the establishment of these new inequalities, Pečarić and Perić make a valuable contribution to the development of mathematical tools for studying functions on time scales, thereby reinforcing the theoretical foundation of this evolving field.

Wang and Sun (2012) wang and Sun examine solutions that produce positive outcomes for higher-order multi-point boundary value problems defined on time scales. Their study focuses on a class of boundary value problems involving higher-order dynamic equations, incorporating multi-point boundary conditions to extend the scope of existing research. By doing so, they provide a comprehensive analysis of positive solutions, offering valuable insights into the qualitative behavior of such systems. Their work is instrumental in enhancing our understanding of how solutions to higher-order dynamic equations behave under varied boundary conditions, contributing significantly to the theoretical advancement of time scale boundary value analysis.

Yan (2005) this study primarily focuses on three-point boundary value problems for higher-order dynamic equations on time scales. In this work, Yan investigates a specific class of boundary value problems within the time scale framework and addresses a significant open question in the field by establishing the existence of positive solutions using analytical techniques tailored to this context. The research offers fresh insights into the existence and uniqueness of solutions in higher-order scenarios, thereby reinforcing the theoretical foundations of boundary value problems in the realm of time scale calculus.

Kusano and Naito (2000) this study explores the oscillatory behavior of second-order delay dynamic equations within the framework of time scales. As noted, time scale theory provides a unified structure that encompasses both discrete and continuous calculus. By generalizing classical results from differential equations, the authors investigate how delay effects influence the dynamics of such equations across various temporal domains, offering broader insights into their long-term behavior in this extended mathematical setting.

Zhang (2007) this study focuses on the existence of solutions for higher-order boundary value problems on time scales. Expanding the scope of analysis to include higher-order cases, the author delves into the mathematical properties of dynamic equations within this unified framework. Through the use of rigorous analytical techniques, Zhang establishes the conditions under which such boundary value problems admit solutions. This work not only strengthens the theoretical foundations of dynamic equations on time scales but also provides valuable insights that bridge the gap between abstract theory and practical applications, particularly in the context of higher-order systems.

EXISTENCE AND UNIQUENESS

Compared to IVPs, BVP existence and uniqueness questions are far more challenging. There isn't a broad theory, in fact. On the other hand, for an overview of a range of possible approaches, a substantial body of literature on specific cases exists. Think about the IVP.

$$y'(x) = f(x, y(x)), \;\; y(a) = s \tag{2}$$

matching the Tribute viewed as in (1). The existence of an answer for (1) relies upon the reasonability of the nonlinear system of equations on the off chance that this IVP has an answer for each conceivable starting vector s.

$$g(s, y(b; s)) = 0 \tag{3}$$

given some initial value y(a)=s, the configuration of the IVP (2) evaluated at x=b is y(b;s). If there is a singular arrangement of the nonlinear system g(s,y(b;s))=0, then s is the most noteworthy arrangement of its kind.

For straight BVPs, when the boundary conditions and ordinary differential equations (ODEs) are both direct, the condition g(s,y(b;s))=0 is a straight system of logarithmic equations. In most cases, there will be either zero, one, or an infinite number of solutions when dealing with systems of straight logarithmic equations.

Finite arrangements are one more opportunities for nonlinear issues, notwithstanding the decisions for straight problems. Look at the accompanying fundamental shot movement model with air obstruction:

$$\begin{aligned} y' &= \tan(\phi), \\ v' &= -\frac{g}{v}\tan(\phi) - \nu v \sec(\phi), \\ \phi' &= -\frac{g}{v^2}\,. \end{aligned} \tag{4}$$

It is feasible to consider these equations addressing the planar movement of a shot from a gun. For this situation, y means the shot's level over the cannon's level, v signifies its speed andʋ indicates the shot's point of direction regarding the flat. The separation from the gun estimated on a level plane is demonstrated by the free factor x. Air opposition, or rubbing, is addressed by the consistent v, while the reasonably scaled gravitational steady is signified by g. Three-layered impacts like crosswinds and shot revolution are disregarded by this model. The cannon's underlying level is y(0)=0 andits gag speed, v(0), is fixed.

NUMERICAL METHODS

As previously mentioned, traditional shooting methods used for solving boundary value problems (BVPs) often face inherent limitations. These issues can be partially addressed through variations of the shooting technique, commonly referred to as multiple shooting methods, which improve solution stability and accuracy. Most general-purpose BVP solvers are built upon global methods, typically classified into two main categories. The first is the finite difference method, which involves constructing a mesh over the interval [a,b][a, b][a,b] and replacing the differential terms with finite difference approximations at each mesh point. By incorporating the boundary conditions into these finite difference equations, a system of algebraic equations is formed. While this system is often nonlinear, it becomes linear if both the differential equations and boundary conditions are linear. These solvers usually utilize local error estimates—based on higher-order difference methods, such as deferred correction—to adapt the mesh and satisfy a user-specified error tolerance. The second global approach involves constructing an approximate solution using a basis from a linear function space, typically defined piecewise over the mesh. In collocation methods, this approximate solution is substituted into the differential system andthe system is enforced to hold exactly at selected collocation points. To ensure solvability, the number of unknown coefficients in the approximation must match the total number of collocation conditions and boundary conditions, effectively equal to the dimension of the chosen function space. A common and effective choice for such approximations is a space of piecewise polynomial splines. Achieving optimal accuracy requires careful selection of collocation points within the mesh. As with finite difference methods, adaptive mesh refinement is employed, guided by local error estimates derived from approximations at different levels of precision. This process allows the solver to adjust the mesh spacing dynamically to maintain high accuracy and computational efficiency.

CONCLUSION

This study explores a critical yet underexplored area of research—higher-order boundary value problems (BVPs) for delay differential equations—with significant implications for control theory, physics andapplied mathematics. While considerable progress has been made in understanding lower-order BVPs with delays, this work emphasizes the importance and complexity of their higher-order counterparts. The research focuses on the explicit first-order system representation and the formulation of multi-point higher-order BVPs for ordinary differential equations. It further investigates the intricacies of two-point and multi-point boundary conditions, highlighting the necessity of independent boundary constraints and the occasional requirement for system reformulation to enhance computational efficiency. The study also addresses the fundamental questions of existence and uniqueness of solutions, acknowledging the theoretical complexity involved. In addition, it evaluates

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global approaches such as finite difference methods and collocation techniques, alongside numerical strategies like shooting methods. The findings make it clear that solving higher-order BVPs with delays involves substantial theoretical and computational challenges, underscoring the need for continued research and the development of more advanced numerical methods to effectively address these complex problems.

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