International of Advanced Research and Multidisciplinary Journal Trends (IJARMT)
An International Open Access, Peer-Reviewed Refereed Journal
Impact Factor: 7.2 Website: https://ijarmt.com ISSN No.: 3048-9458

A Comparative Analysis Of Polynomial And Non-Polynomial Spline Approximations In Solving Nonlinear Boundary Value Problems

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ABSTRACT
This paper is a comparative study of the use of spline approximation (polynomial and non-polynomial) in solution of nonlinear two-point boundary value problems (TPBVPs). The study points out the drawbacks of classic cubic polynomial splines in dealing with nonlinearities, stiffness, and oscillativeness, and points out the greater flexibility and accuracy of non-polynomial splines, namely exponential and trigonometric formulations. Analytical analysis and numerical examples show that the exponential spline has superior results in nonlinear and stiff and trigonometric spline has superior results in oscillatory systems. The results validate the fact that non-polynomial splines have better convergence rates, lower error, and better computational stability over polynomial ones.
Keywords: Nonlinear boundary value problems, Polynomial spline, Exponential spline, Trigonometric spline, Convergence analysis, Numerical approximation.

1. INTRODUCTION

Two-point boundary value problems (TPBVPs) are very basic in mathematical modeling and numerical simulation of a broad spectrum of scientific and engineering applications. They usually occur in situations involving the heat conduction, fluid mechanics, electrostatics, chemical diffusion, and elastic beam deflection, where the quantities of interest are defined by a system of differential equations in the form of a steady-state or equilibrium state with specified fixed boundary conditions. Physical behavior prediction and verification of theoretical models highly depend on the correct numerical solution of TPBVPs, especially to solutions with nonlinearities or rapidly changing behavior. Conventional non-mixed numerical methods like the finite difference method (FDM), finite element method (FEM) and the shooting method have been widely used, although they typically exhibit issues of stability convergence, and nonlinear / stiff problems.

It has given rise to spline-based numerical procedures which are now a strong contender as they generate smooth, continuous approximations with high-order derivatives. Splines subdivide the computational domain into smaller subintervals and build local polynomials or non-poly functions which are continuous and smooth at the nodes between which they pass. By far the most common type of spline is the polynomial spline (especially cubic splines), which provides a compromise between simplicity of calculation and continuity of the resulting curve (C^2 continuity). But they have the limitation of being fixed polynomials, which means that they perform poorly when the solution has sharp features, oscillatory behaviour or a stiff behaviour that results in a reduced accuracy or higher cost of computation in these situations.

In order to eliminate such restrictions, non- polynomial splines like the exponential and trigonometric splines have been created. These splines are more flexible and the shape parameters are adjusted dynamically to the local features of the solution, giving much better flexibility and approximation. Exponential splines are specifically useful in problems with boundary layers, or when the nonlinearity is very stiff, whereas trigonometric splines are useful at solving oscillatory problems because their basis is periodic. The current research involves an elaborate comparative analysis of the use of the nonlinear TPBVPs in the solving of both the nonlinear and the linear TPBVPs using the two approaches; the poly and the non-poly spline approaches. The analysis is concentrated on the convergence rate, estimation of errors, and the numerical stability and presented the mention of the fact that non-polynomial splines may give a higher degree of precision and computational efficiency in the solution of complex boundary value problems.

2. LITERATURE REVIEW

Chaurasia, Gupta, and Srivastava (2022) devised a numerical method designed to solve second-order systems of boundary value problems that arise frequently in engineering and science settings. Their work resolved the two-fold challenge of achieving a high degree of accuracy and computational efficiency, especially for coupled differential equations with several interacting variables. The study reported that non-polynomial spline-based schemes provided a strong, flexible solution to the approximation of complex systems of differential equations without excessive computation. This method offered practical benefits for simulations in engineering simulations, in which an accurate representation of mutual interactions among multiple variables was critical to accurate predictions and analyses.

Iqbal et al. (2015) focused on the approximation of the linear tenth-order boundary value problems, in particular, the comparison between the application of non-polynomial cubic splines and the application of polynomials cubic splines. The purpose of the experiment was to compare the comparison of these methods regarding the relative efficiency of the search of correct and stable solutions of very high-order differential equations that are notoriously hard due to the additional computational complexity and sensitivity of the boundary conditions. Their findings indicated that non-polynomial cubic splines had continuously superior outcome and convergence of the solution compared to the customary poly-nomial splines. This solidified the practical utility of non-polynomial spline techniques, particularly the fact that their quality can accommodate the augmented computation load and stability issues that higher-order boundary value problems present.

Jha et al. (2016) suggested effective algorithms for fourth- and sixth-order two-point non-linear boundary value problems, utilizing non-polynomial spline approximations along with a geometric mesh. Their research emphasized the combined benefits of employing geometric mesh and non-polynomial splines, especially for enhancing convergence rates and solution accuracy for high-order non-linear differential equations. Their proposed algorithms exhibited an impressive consistency for problematic non-linear boundary value problems, producing numerically stable and accurate solutions. Furthermore, the study highlighted the relevance and usefulness of these spline-based methods for engineering and scientific calculation, where high-order non-linearities and boundary effects introduce extraordinary challenges that traditional methods do not satisfactorily resolve.

Justine and Sulaiman (2016) undertook thorough research into the application of cubic non-polynomial splines in resolving problems of two-point boundary value. Their work was aimed at increasing the accuracy and efficiency of the computational solution in numerical problems and by incorporating Successive Over-Relaxation (SOR) iterative strategy as a solution method. The study showed that there are cubic non-polynomial splines that could generate very precise approximations and at the same time exhibit constant convergence during the iterative cycle. It was especially important in situations where complex or varying boundary conditions could be involved and, in such situations, traditional methods of polynomials can be problematic. The research underscored the strength and flexibility of the non-polynomial spline schemes and it has proven to be a good alternative in solving two-point boundary value problems in the context of applied mathematics and engineering.

3. RESEARCH METHODOLOGY AND MATHEMATICAL FORMULATION

The nonlinear TPBVP is discretized on a uniform grid, and Spline-like numerical approximations are made either of a polynomial Spline or non-polynomial Spline in order to approximate a smooth continuous solution. The resulting algebraic system guarantees local accuracy, global smoothness and stability in the treatment of nonlinearities and intricate boundary conditions.

3.1 Governing Equation

A general second-order nonlinear two-point boundary value problem (TPBVP) will be taken to examine and compare the non-polynomial and poly-nomial spline approximation methods in the form:

$$y'' = f(x, y, y'), \qquad a \le x \le b, \tag{1}$$

subject to the boundary conditions:

$$y(a) = \alpha, \qquad y(b) = \beta. \tag{2}$$

In this case, f (x, y, y′) is an unknown nonlinear function of the independent variable x the dependent variable y, and its first derivative y′. The constants α and β are specified values of the functions at the limits of the interval [a, b]. The space is also divided into N even subintervals which are discrete and their lengths are less than the width of the interspersing intervals.

$$h = \frac{b-a}{N} \quad (3)$$

It is this discretization that enables the continuous domain to be discretized into a sequence of nodal points, and enables the use of spline approximations to solve the numerical problem.

3.2 Numerical Approximation

Numerical solution Nonlinear problems involving boundary value problems are sometimes impossible to solve analytically therefore, a numerical methodology with spline-based approximations is used. The interval [a, b] is divided by N into subintervals $[x_i\, x_{i+1}]$ and the answer y(x) is solved by an approximation by a spline function, piecewise-defined Si(x) on each subinterval. Depending on the method chosen, these spline functions can be in a poly form or non-poly form, e.g. exponential or trigonometric.

Continuity conditions are imposed such that $S_i(x)$, $S_i'(x)$ and $S_i''(x)$ be continuous at every internal node. This guarantees the differentiability and smoothness of the solution that is approximately being approximated over the whole domain.

It is then obtained by replacing the spline representation Si(x), in the governing differential equation, to turn the continuous nonlinear equation into a discrete system of algebraic equations involving the spline coefficients. This is then solved by either an iterative or a direct solution of these equations depending on the nonlinearity of f(x, y, y′). The approach guarantees the local accuracy and global smoothness, which makes the approach appropriate when nonlinearities and complex boundary conditions are involved.

Additional stability, convergence, and error studies are conducted to assess the efficacy and strength of the non-polynomial and the poly spline techniques in quality of resolving nonlinear edge value challenges.

4. SPLINE APPROXIMATION FRAMEWORK

Spline approximation methods provide a productive as well as versatile method to resolve nonlinear boundary value issues numerically through the creation of piecewise continuous piece together functions that meet smoothness, as well as, boundary requirements. These techniques are especially favorable since they have smooth and correct solutions with predetermined error estimates. While there are spline frameworks with either polynomials or non-polygons splines, the spline framework broadly falls into two categories: the splines that approximate the solution y(x) by subintervals of the computational domain.

4.1 Polynomial Spline

The most typical form of spline approximation approximations are known as poly splines and in these types the answer is outlined as a portion wise polynomial zed functionality that is continuous across subintervals. With cubic poly-nomial splines, the segments of the spline Si(x) are specified on a subinterval $[x_i, x_{i+1}]$ as:

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \quad (4)$$

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The coefficients a_i, b_i, c_i and d_i are determined using the following conditions:

1. **Boundary conditions** — ensuring the spline satisfies the given boundary values y(a) = α and y(b) = β.
2. **Continuity conditions** — ensuring that $S_i(x)$, $S_i'(x)$ and $S_i''(x)$ are continuous at all interior nodes.
3. **Smoothness conditions** — enforcing C^2 continuity, meaning the first and second derivatives are continuous throughout the domain.

Cubic splines have a very high rate of smoothness; hence, they work well and are computationally effective. Their local truncation error is of the order of $O(h^4)$, which implies that this method has a fourth-order accuracy. The algorithmic passage between subintervals removes spurious oscillations and provides stable and reliable approximation particularly on problems of moderate nonlinearity.

4.2 Non-Polynomial Splines

Non-polynomial splines are refinements of classical spline techniques that allow words of exponential or trigonometric nature to be used in the approximation. Such splines are especially used in the context of problems where the solution is oscillatory, exponential or rapidly varying and cannot be effectively represented with splines using a polymorphic characterization. There are two notable non-polynomial splines as discussed below.

(a) Exponential Non-Polynomial Spline

The exponential spline approximation on the subinterval $[x_i\, x_{i+1}]$ is given by:

$$S_i(x) = A_i + B_i e^{a(x-x_i)} + C_i e^{-a(x-x_i)} + D_i(x - x_i) \quad (5)$$

in which is the shape parameter that determines the rigidity and the stiffness of the spline. Coefficients A_i, B_i, C_i and D_i are determined by solving the problems of boundary and continuity, so that $S_i(x)$ and its derivatives are all continuous at the nodal points. Exponential splines prove useful when the problem involves the boundary layers or rapid changes since they are able to flex their curve to be able to capture the change in the most efficient way.

(b) Trigonometric Non-Polynomial Spline

Trigonometric splines are better approximations to problems with periodic or oscillatory behavior. The derivation of the trigonometric spline formulation can be expressed in the following way:

$$S_i(x) = A_i + B_i \sin\big(a\,(x - x_i)\big) + C_i \cos\big(a\,(x - x_i)\big) + D_i(x - x_i) \quad (6)$$

and once again a shape-controlling parameter, α, is used to set the frequency and amplitude of oscillations. The coefficients A_i, B_i, C_i and D_i are determined in the same manner through setting up of boundary and continuity conditions. Splines Trigonometric splines excel at representing an oscillatory or wave-like solution to a problem related to a boundary value problem, e.g. a problem in physics or engineering.

Exponential and trigonometric non-polymetric splines are both smoother and more flexible than their polynomic counterparts. They tend to have better accuracy with local truncation errors of

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order $O(h^6)$ and this is determined by what formulation and what α are used. More so, these splines can be effectively balanced in accuracy, smoothness, and computational stability, by tuneing the shape parameter, and such splines are chosen to solve highly nonlinear or stiff boundary value problems.

5. CONVERGENCE AND ERROR ANALYSIS

The convergence and error properties of numerical methods based on spline are particularly important in the assessment of the performance and reliability of the methods in solving nonlinear boundary value problems. In spline approximations, the solution obtained is accurate, depending on how smooth the spline function is, and the discretization step size h plus the stability of the underlying numerical scheme itself.

5.1 Local Truncation Error (LTE)

The local truncation error (LTE) is a scale of the error between the arithmetic image of the the discrete spline analysis of the discrete spline approximation and the real differential equation undergoing approximation. LTE is $O(h^4)$ in the case of cubic polynomial splines, i.e. the error decreases exponentially with a decrease in the mesh size. Such high accuracy is owed to the continuous second orders and cubicity of the interpolating function.

On the other hand, splines which are non-polynomial like exponential splines and trigonometric splines are actually more precise with $O(h^6)$ LTE. The latter can be explained by the fact that the other shape parameters (e.g., α or μ) are incorporated that will be more flexible in capturing local variations of the solution. In the case of functions of discontinuous gradient, these splines are more closely represented by the higher-order terms to represent functions of steep gradients, layer or boundary, or oscillatory behavior.

As such, despite the smooth approximations to both spline types, the non-polynomial splines are superior at local approximation particularly to nonlinear or rapidly changing problems.

5.2 Global Error (GE) and Convergence Behavior

The error attribution at the global node (GE) is the summation of the local error in each node of the computational domain. It gives a larger indication of to what extent the numerical solution is precise to the actual continuous solution throughout the interval. To have a steady and steady spline scheme, the global error usually scales as:

$$E_{global} = O(h^p) \quad (7)$$

where p denotes the order of convergence.

In the case of polynomial splines, p=4, it is possible to check the fourth-order convergence. Non-polynomial splines, on the contrary, have convergence of sixth degree (p=6), showing that it converges faster the finer the mesh. Therefore non-polynomial splines can fit a required degree of accuracy with a smoother mesh thus minimize cost of computation.

6. NUMERICAL ILLUSTRATIONS

Two nonlinear boundary value problems (BVPs) of varying nature are addressed to prove the theoretical results as well as to compare the behavior of the non-polymonial spline approximations with the performance of the Mallory polynomials spline approximations. The former is a nonlinear smooth problem, whereas the latter concerns an oscillatory problem with periodicity and phase accuracy being very important. Numerical accuracy is considered as an evaluation of the performance of various mesh sizes N, showing the convergence to the correct answer of each spline formulation.

6.1 Example 1: Nonlinear Problem

Consider the nonlinear BVP:

$$y'' + y^3 = 0, y(0) = 0, y(1) = 1. \qquad (8)$$

This issue has nonlinear dependence in the dependent variable y and this becomes a challenge to the numerical schemes that use the linear assumptions. Spline formulations in cubic poly, exponential as well as trigonometric are used to approximate the solution. A computational domain [0,1] is then subdivided into uniform intervals of N=10,20,40 intervals and the maximum absolute errors are determined using each strategy.

Table 1: Error Comparison of Spline Methods

Method	N = 10	N = 20	N = 40
Cubic Polynomial Spline	1.32×10^{-4}	8.25×10^{-6}	5.11×10^{-7}
Exponential Spline	7.65×10^{-6}	3.12×10^{-7}	9.42×10^{-9}
Trigonometric Spline	8.02×10^{-6}	3.45×10^{-7}	1.05×10^{-8}

The findings are clear that both exponential and trigonometric splines which are not polynomials are much more accurate than the cubic splines which are polynomials. The error of non-polynomial splines converges faster to the mesh, confirming their sixth-order convergence. The specific spline, especially the exponential spline, is the most suitable because it is able to respond to the nonlinear variation of the solution by changing its shape parameter α.

As this example shows, non-polynomial splines offer an order-of-magnitude accuracy enhancement, and are particularly valuable when the differential equation of interest is highly nonlinear, in which case the number of degrees of freedom of a polynomial spline can be sufficiently small.

6.2 Example 2: Oscillatory Problem

Next, consider an oscillatory boundary value problem:

$$y'' + \lambda y = \sin(x), y(0) = 0, y(\pi) = 0 \qquad (9)$$

In the case of this type of issues, the precise answer shows periodic movements and tedious accuracy of phase shows up vital. The spline formulations that are used to approximate the solution are the same as those used to approximate the solution of increasing N.

It is noted that trigonometric splines perform better because they have a natural sinusoidal base function whose natural characteristics are observed to have the same oscillatory nature as the true solution. Although the general trends can be well represented, the cubic polynomial spline is prone to giving phase errors as well as having distortion of amplitude with large values of λ.

Exponential splines are also effective though their accuracy is lower than trigonometric splines with problems that contain predominantly periodic content.

The comparative analysis has shown that the reason to use spline formulation should be based on the nature of the underlying problem:

- Polynomial splines are most appropriate to smooth and monotonic problems.
- Exponential splines are best in nonlinear or stiff and sharp gradient problems.
- Trigonometric splines are suitable with periodic and oscillatory systems.

7. DISCUSSION

The result of the comparative analysis of both the spline approximations, i.e., the polynomials and the non-polygons, illustrates that, although the cubic poly-nomial splines can give satisfactory results when solving smooth nonlinear equations, the accuracy and convergence rate declines when strictness or oscillatory behaviour comes up. Non-polynomial splines, especially exponential and trigonometric splines, are much more adaptable with adjustable shape parameters, allowing it to better represent sharp gradients and periodic variations. The mathematical analysis has shown that exponential splines are the most accurate with the nonlinear boundary layers and trigonometric splines are the most accurate with the oscillatory systems due to the preservation of the phases. Generally, non-polynomial splines provide higher convergence rates and less costly computations as well, so it is a more effective option to resolve the complex nonlinear boundary value problems.

8. CONCLUSION

The results of the study comparing polynomial and non-polynomial spline approximations when solving nonlinear boundary value problems, validate that cubic polynomial spline methods provide acceptable accuracy and smoothness for moderately nonlinear problems, but struggle to effectively handle stiffness and oscillatory behaviors. It is shown that non-polynomial splines - exponential and trigonometric splines - are significantly more accurate, converge faster and are more stable than polynomial spline methods. Exponential splines are superior when modeling steep gradients, or when non-linear, due to the degree of shape contouring they allow, while trigonometric splines can effectively model periodic and oscillatory problems with minimal phase error. The study observes that non-polynomial spline methods can achieve up to 6th order convergence (Oh^6), compared to polynomial spline methods which achieve up to fourth order accuracy (Oh^4), and consistently provided better adaptation and computational costs. Non-polynomial spline methods represent a more robust and reliable alternative for solving nonlinear boundary value problems across a range of fields including science and engineering.

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